ALGEBRA 2

RATIONAL EXPONENTS

LEARNING ACTIVITIES PACKAGE

BEHAVIORAL OBJECTIVES

Upon completion of this L.A.P., the student will be able to

I. Given an expression of the form $\sqrt[n]{ab}$, identify

- A. The root index
- B. The radicand
- C. The radical

II. Given an expression of the form $a^{p/r}$ identify

- A. The role of p
- B. The role of r

III. Given an expression containing a radical, rewrite the expression using rational exponents.

IV. Given an expression containing rational exponents, rewrite the expression using a radical.

V. Simplify a given radical expression by

- A. Removing all factors which have powers greater than or equal to the root index from under the radical
- B. Allowing only integer coefficients in the radicand
- C. Removing all radical expressions from the denominator

VI. Rationalize the denominator of an expression when the denominator is of the form

- A. $a^{p/r}$ or $\sqrt[r]{a^p}$
- B. $\sqrt{a} \pm b$ or $\sqrt{a} \pm \sqrt{b}$

VII. Perform the common operations of multiplication, division, and raising to a power on given radical expressions

VIII. Solve equations which contain one or two radical expressians

IT IS IN THIS L.A.P. THAT YOU MIGHT SAY
WE GET TO THE ROOT OF THE PROBLEM!

SECTION I RADICALS AND RATIONAL EXPONENTS

In this L.A.P. we are going to study exponential expressions where the exponents are rational numbers.

REMEMBER: A rational number is one that can be expressed as the quotient of two integers, $\frac{a}{b}$, where $b \neq 0$. (In other words, a rational number is a fraction.) Why do we say $b \neq 0$?

EXAMPLES OF RATIONAL NUMBERS: $\frac{1}{2}, \frac{7}{9}, \frac{5}{3}, 6, 127, 0.$

$x^{\frac{1}{3}}$: Base: x Exponent: $\frac{1}{3}$

$$\left(x^{\frac{1}{3}}\right)^3 = x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x \qquad (\sqrt[3]{x})^3 = x$$

$x^{\frac{1}{3}}$ means the same as $\sqrt[3]{x}$ (read the "cube root of x")

When a rational number, $\frac{a}{b}$, is used as an exponent, each integer has its own function.

> The DENOMINATOR of the rational exponent gives us the root of the expression.

EXAMPLE: $(3ab)^{\frac{1}{6}} = \sqrt[6]{3ab}$ (Read "the sixth root of 3ab")

Definitions: "6" is the root index

"3ab" is the radicand (That is what is under the radical.)

$\sqrt{\ }$ is the radical

Note: When the radical contains no root index, it is understood to be 2.

$\sqrt{7x} = (7x)^{\frac{1}{2}}$ (1)

$7\sqrt{x} = 7x^{\frac{1}{2}}$ (2) Sutdy both (1) and (2). They are different. Be careful!

> The NUMERATOR of the rational exponent gives us the power to which the base must be raised.

EXAMPLE: $a^{\frac{2}{3}} = \sqrt[3]{a^2}$ or $a^{\frac{2}{3}} = (\sqrt[3]{a})^2$

$$(2xy^2)^{\frac{5}{7}} = \sqrt[7]{(2xy^2)^5} \text{ or } (2xy^2)^{\frac{5}{7}} = (\sqrt[7]{2xy^2})^5$$

Let's do an example together:

$$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{54} = 4$$

STOP: Let's talk about an easier way:

In this case, as in many, it would have been easier if we had taken the cube root of 8 before we squared 8 to got the 64. If you get into the habit right away of expressing any base as a possible power, for example $8 = 2^3$, you will save yourself a lot of work and your answers have a better chance of being right.

In other words, use this SHORT CUT:

$$8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}$$ (Remember: to raise a power to a power, we multiply exponents.)

$$(2^3)^{\frac{2}{3}} = 2^{3(\frac{2}{3})} = 2^2 = 4$$

The value of this method can be seen in examples like these:

Evaluate	Long way	Easy way
$16^{\frac{3}{4}}$	$16^{\frac{3}{4}} = \sqrt[4]{16^3} = \sqrt[4]{4096} = ?$	$16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^3 = 8$
$32^{\frac{2}{5}}$	$32^{\frac{2}{5}} = \sqrt[5]{32^2} = \sqrt[5]{1024} = ?$	$32^{\frac{2}{5}} = (2^5)^{\frac{2}{5}} = 2^2 = 4$

Now that you know the easy way of doing these, try our first exercise.

EXERCISE 1

1. Complete the following chart so that you will have a reference of some of the "handy" numbers.

$2^2 =$ 4	$3^2 =$ ____	$5^2 =$ ____	$7^2 =$ ____
$2^3 =$ ____	$3^3 =$ ____	$5^3 =$ ____	$7^3 =$ ____
$2^4 =$ ____	$3^4 =$ ____	$5^4 =$ ____	$11^2 =$ ____
$2^5 =$ ____	$3^5 =$ ____	$6^2 =$ ____	$12^2 =$ ____
$2^6 =$ ____		$6^3 =$ ____	$13^2 =$ ____
$2^7 =$ ____			$14^2 =$ ____
$2^8 =$ ____			$15^2 =$ ____

2. Evaluate each of the following:

a. $25^{\frac{1}{2}}$ = ____ b. $8^{\frac{1}{3}}$ = ____ c. $64^{\frac{1}{6}}$ = ____ d. $125^{\frac{2}{3}}$ = ____

e. $81^{\frac{3}{4}}$ = ____ f. $-32^{\frac{3}{5}}$ = ____ g. $100^{\frac{3}{2}}$ = ____ h. $-49^{\frac{1}{2}}$ = ____

i. $(-81)^{\frac{1}{2}}$ = ____ j. $(-27)^{\frac{1}{3}}$ = ____ k. $10,000,000^{\frac{3}{7}}$ = ____

l. $16^{\frac{5}{4}}$ = ____ *m. $100^{-\frac{1}{2}}$ = ____ n. $32^{-\frac{3}{5}}$ = ____ o. $81^{-\frac{3}{4}}$ = ____

*Remember: a <u>negative exponent</u> says; <u>"take the reciprocal."</u>

$$8^{-\frac{1}{3}} = \frac{1}{8^{1/3}} = \frac{1}{(2^3)^{1/3}} = \frac{1}{2}$$

3. Rewrite each of the following using radicals instead of rational exponents:

a. $x^{\frac{1}{3}}$ = ______ b. $y^{\frac{2}{3}}$ = ______ c. $w^{\frac{1}{2}}$ = ______ d. $2a^{\frac{3}{4}}$ = ______

e. $(2a)^{\frac{3}{4}}$ = ____ f. $2^{\frac{1}{4}} a^{\frac{3}{4}}$ = ____ g. $(6x^2)^{\frac{1}{3}}$ = ______ h. $a^{\frac{1}{5}} b^{\frac{2}{5}}$ = ______

i. $2x^{\frac{1}{2}}y^{\frac{1}{2}}$ = ____ j. $a^{\frac{5}{6}} b^{\frac{1}{6}} c$ = ____ k. $x\ y\ z^{\frac{1}{4}}$ = ____ l. $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ = ______

m. $x^{\frac{1}{3}} - y^{\frac{1}{3}}$ = ____ n. $6a^{\frac{1}{2}} + 7b^{\frac{1}{2}}$ = ______

4. Rewrite each of the following using rational exponents instead of radicals:

a. $\sqrt{w}$ = ____ b. $\sqrt[3]{xy}$ = ____ c. $\sqrt[5]{a^2b^3}$ = ____ d. $\sqrt[4]{(pq)^2}$ = ____

e. $\sqrt[5]{pq^2}$ = ____ f. $\sqrt[7]{5w^2}$ = ____ g. $\sqrt[7]{(5w)^2}$ = ____ h. $5\sqrt[7]{w^2}$ = ____

i. $w^2\sqrt[7]{5}$ = ____ j. $a\sqrt{bc}$ = ____ k. $\sqrt[3]{aw^5}$ = ____ l. $\sqrt[7]{x^2y^3z^9}$ = ____

m. $\sqrt[6]{a^2b^3}$ = ____

SECTION II

SIMPLIFYING RADICAL EXPRESSIONS

Some basic information about real numbers and radicals:

(1) For every real number 'a', and for every positive even integer 'n',

$$\sqrt[n]{a^n} = |a|$$ (The answer to an even root problem is never negative.)

Examples: $\sqrt{(-2)^2} = 2$; $\sqrt{x^6} = |x^3|$

(2) In the set of real numbers it is not possible to take even roots of negative numbers.

Examples: $\sqrt{-4}$ = no real number solution

$\sqrt{x^3} = x\sqrt{x}$ if and only if x is greater than or equal to 0.

Throughout this L.A.P., assume that all radicands are properly defined to permit solutions. Also, you may omit absolute value signs from the solutions. Thank the management for this freedom at your first opportunity.

Now, down to work! When we simplify a radical expression we are trying to remove as much as possible from under the radical. In order for an expression to be in simplified form, the following three conditions must be satisfied:

1. No integer or polynomial in the radicand can have a power that is the same or higher than the root index.

Example 1: $\sqrt[3]{125x}$ is not simplified because $\sqrt[3]{125x} = \sqrt[3]{5^3 x}$ and the exponent '3' is the same as the root index '3'.

$$\sqrt[3]{125x} = \sqrt[3]{5^3 x} = \sqrt[3]{5^3} \cdot \sqrt[3]{x} = 5^{3/3} \cdot x^{1/3} = 5\sqrt[3]{x}$$

Example 2: $\sqrt{x^3} = \sqrt{x^2 \cdot x} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}$

Example 3: $\sqrt[7]{x^9}$

As in most advanced algebra problems, there are several ways you can begin to simplify a radical expression, but you will save yourself a lot of work and have a better chance at the correct answer, if you follow the general steps shown at the right.

(a) Change to rational exponents: $\sqrt[7]{x^9} = x^{\frac{9}{7}}$.

(b) Change $\frac{9}{7}$ to a mixed number: $x^{\frac{9}{7}} = x^{1\frac{2}{7}}$.

(c) Remember your rule for multiplying factors of the same base, by adding exponents, the following is true:

$$x^{1\frac{2}{7}} = x^1 \cdot x^{\frac{2}{7}}$$

TURN THE PAGE FOR THE EXCITING CONCLUSION!

(d) Any exponent that is an integer can be removed from the radical. The base that has a fraction for an exponent must be put back under a radical.

$$x^1 \cdot x^{2/7} = x\sqrt[7]{x^2}$$

The expression is now simplified.

Example 4: $\sqrt[3]{27\,x^4\,y^6} = \sqrt[3]{3^3\,x^4\,y^6}$

$= 3^{3/3}\,x^{4/3}\,y^{6/3}$

$= 3 \cdot x^{1\frac{1}{3}} \cdot y^2$

$= 3 \cdot x^1\,x^{1/3}\,y^2$ (Bring the bases with integers as exponents to the left.)

$= 3\,x\,y^2\,x^{1/3}$

$$\sqrt[3]{27\,x^4\,y^6} = 3\,x\,y^2\sqrt[3]{x}$$

NOTE: The little "trick" of expressing any number in the radicand in its prime fractorization helps you see immediately if any numerical factor has a power greater than or equal to the root index.

You might not simplify $\sqrt[3]{16}$, but you will see that $\sqrt[3]{2^4}$ can be simplified. That '4' as a power should be a "red flag" when the root index is 3.

2. There can be no fractions nor bases having negative exponents under the radical.

Example 1: $\sqrt{3\,x^{-2}} = 3^{1/2}\,x^{-2/2} = 3^{1/2}\,x^{-1} = \dfrac{\sqrt{3}}{x}$

Example 2: $\sqrt[3]{x\,y^{-6}} = x^{1/3}\,y^{-6/3} = x^{1/3}\,y^{-2} = \dfrac{\sqrt[3]{x}}{y^2}$

Example 3: $\sqrt{\dfrac{5}{x}} = \dfrac{\sqrt{5}}{\sqrt{x}}$, which brings us to the next rule:

3. There can be no radical in the denominator.

Since in example 3 above, we have $\sqrt{x}$ in the denominator, we must rationalize the denominator. This means we have to get rid of the radical in the denominator.

Remember: $\dfrac{\sqrt{5}}{\sqrt{x}} \cdot \dfrac{\sqrt{x}}{\sqrt{x}} = \dfrac{\sqrt{5\,x}}{x}$

Now, the problem given in Example 3 is simplified.

SECTION II PART 2 RATIONALIZING A DENOMINATOR

Example 1:

$$\sqrt{\frac{7a^{-1}}{w^3}} = \sqrt{\frac{7}{a\,w^3}} = \frac{7^{\frac{1}{2}}}{a^{1/2}\,w^{3/2}} =$$

To complete this problem we need to multiply both numerator and denominator by a factor which will leave only positive whole numbers as exponents in the denominator.

$$= \frac{7^{\frac{1}{2}}}{a^{1/2}\,w^{3/2}} \cdot \frac{a^{1/2}\,w^{1/2}}{a^{1/2}\,w^{1/2}} = \frac{7^{1/2}\,a^{1/2}\,w^{1/2}}{a\,w^2}$$

$$\sqrt{\frac{7a^{-1}}{w^3}} = \frac{\sqrt{7\,a\,w}}{a\,w^2}$$

Example 2:

$$\sqrt[3]{x\,y^{-2}} = x^{1/3}\,y^{-2/3} = \frac{x^{1/3}}{y^{2/3}}$$

In order to have the denominator to have a positive whole number as an exponent we should multiply both numerator and denominator by $y^{1/3}$

$$= \frac{x^{1/3}}{y^{2/3}} \cdot \frac{y^{1/3}}{y^{1/3}} = \frac{x^{1/3}\,y^{1/3}}{y^{3/3}}$$

$$\sqrt[3]{x\,y^{-2}} = \frac{\sqrt[3]{x\,y}}{y}$$

Example 3:

$$\sqrt[5]{\frac{2x^{-2}}{y^3\,z}} = \sqrt[5]{\frac{2}{x^2\,y^3\,z}} = \frac{2^{1/5}}{x^{2/5}y^{3/5}z^{1/5}}$$

What shall we choose to rationalize the denominator? (Ask yourself: What fraction when added to each fractional exponent in the denominator will bring it up to an integer?)

$x^{2/5}$ calls for $x^{3/5}$

$y^{3/5}$ calls for $y^{2/5}$

$z^{1/5}$ calls for $z^{4/5}$

So we have:

$$\frac{2^{1/5}}{x^{2/5}y^{3/5}z^{1/5}} \cdot \left(\frac{x^{3/5}y^{2/5}z^{4/5}}{x^{3/5}y^{2/5}z^{4/5}}\right) = \frac{2^{1/5}x^{3/5}y^{2/5}z^{4/5}}{x^{5/5}y^{5/5}z^{5/5}} = \frac{\sqrt[5]{2x^3y^2z^4}}{xyz}$$

Before you attempt EXERCISE 2, study the additional examples of simplification given on page 8;

4. $16^{12/5} = 2^{4(12/5)} = 2^{48/5}$

$= 2^{9\frac{3}{5}} = 2^9 \cdot 2^{3/5} = \underline{2^9 \sqrt[5]{2^3}}$

---

5. $\sqrt[5]{625^3} = \sqrt[5]{(5^4)^3} = \sqrt[5]{5^{12}}$

$= 5^{12/5} = 5^{2\frac{2}{5}}$

$= 5^2 \cdot 5^{2/5} = \underline{25 \sqrt[5]{25}}$

---

6. $\sqrt[3]{\left(\frac{1000}{27}\right)^2} = \sqrt[3]{\frac{(10^3)^2}{(3^3)^2}} = \sqrt[3]{\frac{10^6}{3^6}} = \frac{10^{6/3}}{3^{6/3}} = \frac{10^2}{3^2} = \underline{\frac{100}{9}}$

---

7. $20^{5/2} = (2^2 \cdot 5)^{5/2} = 2^5 \cdot 5^{5/2}$

$= 2^5 \cdot 5^{2\frac{1}{2}} = 2^5 \cdot 5^2 \cdot 5^{\frac{1}{2}}$

$= 32 \cdot 25\sqrt{5} = \underline{800\sqrt{5}}$

---

8. $\sqrt[5]{64\, x^7 y^3 z^{10}} = \sqrt[5]{2^6 x^7 y^3 z^{10}} = 2^{6/5} x^{7/5} y^{3/5} z^{10/5}$

$= 2^{1\frac{1}{5}} x^{1\frac{2}{5}} y^{\frac{3}{5}} z^2 = 2 \cdot 2^{1/5} x^1 x^{2/5} y^{3/5} z^2$

$= 2xz^2 \cdot 2^{1/5} x^{2/5} y^{3/5} = \underline{2xz^2 \sqrt[5]{2x^2y^3}}$

---

9. $\sqrt[4]{16\, x^{-3}\, y^{-8}} = \sqrt[4]{\frac{2^4}{x^3\, y^8}} = \frac{2^{4/4}}{x^{3/4} y^{8/4}}$

$= \frac{2}{y^2\, x^{3/4}}$ (Rationalize the denominator)

$= \frac{2}{y^2\, x^{3/4}} \cdot \frac{x^{1/4}}{x^{1/4}} = \frac{2\, x^{1/4}}{y^2\, x} = \underline{\frac{2\sqrt[4]{x}}{y^2\, x}}$

Now it is time for you to try these. Be sure to refer back to examples given on the previous pages for steps in simplifying.

Above all: DO ONE STEP AT A TIME

Write each step down--do not try to keep it in your head!

EXERCISE 2 Simplify each of the following:

1. $\sqrt[3]{625}$ 2. $\sqrt[5]{32x^3y^{10}}$ 3. $\sqrt{49x^2y^3}$

4. $\sqrt[4]{81a^4b^{12}c^7}$ 5. $\sqrt[4]{25}$ 6. $\sqrt[5]{16p^5}$

7. $\sqrt[n]{2^n}$ 8. $\sqrt[3]{8x^{-3}}$ 9. $18^{2/3}$

10. $16^{4/3}$ 11. $\sqrt[6]{9x^4}$ 12. $\sqrt[6]{\frac{625}{4^{24}}}$

13. $\sqrt[4]{625x^5y^8}$ 14. $\sqrt[6]{125x^9}$ 15. $\sqrt[9]{8x^3y^{-9}}$

16. $\sqrt{5x^{-3}}$ 17. $\sqrt[3]{27x^{-2}}$ 18. $\sqrt[5]{\frac{64a^3}{b^2}}$

19. $\sqrt[4]{xy^{-12}}$ 20. $\sqrt{\frac{49}{x}}$ 21. $\sqrt[7]{\frac{3x^{-2}}{y^2z}}$

22. $\sqrt[7]{625^6}$ 23. $\sqrt[4]{\left(\frac{16}{81}\right)^8}$ 24. $\sqrt[3]{81x^6y^{-2}z}$

25. $\sqrt[5]{\frac{64x^{-2}}{x^3y^{-3}}}$ 26. $\sqrt[6]{\frac{8a^{-3}}{a^5b^{-2}}}$ 27. $\sqrt{\frac{200}{xy}}$

28. $\sqrt[n]{5^nx^{3n}}$ 29. $\sqrt[5]{\frac{32w}{y^3z}}$ 30. $\sqrt[3]{\frac{40x^{-3}}{y^2z^{-6}}}$

SECTION III OPERATIONS WITH RADICALS

A. Let us review some rules for operations with exponents.

(1) When we multiply factors with like bases, we add the exponents.

$$2^a \cdot 2^b = 2^{a+b}$$

If the exponents are fractions, remember you must have the same denominators before you can add the exponents.

$$2^{1/3} \cdot 2^{1/4} \cdot 2^{5/6} = 2^{4/12} \cdot 2^{3/12} \cdot 2^{10/12} = 2^{17/12}$$

$$= 2^{1\frac{5}{12}} = 2\sqrt[12]{2^5}$$

(2) When we divide factors with like bases, we subtract the exponents.

(a) $\dfrac{2^a}{2^b} = 2^{a-b}$

(b) $\dfrac{2^{3/4}}{2^{2/5}} = 2^{3/4 - 2/5} = 2^{15/20 - 8/20} = 2^{7/20} = \sqrt[20]{2^7}$

(3) When we raise a power to a power, we multiply the exponents.

(a) $(2^a)^b = 2^{ab}$

(b) $(2^3)^{2/5} = 2^{6/5} = 2^1 \cdot 2^{1/5} = 2\sqrt[5]{2}$

(c) $(x^{2/3})^{4/5} = x^{8/15} = \sqrt[15]{x^8}$

(d) $\sqrt[3]{\sqrt{\sqrt{5}}} = \left[(5^{1/2})^{1/2}\right]^{1/3} = 5^{1/12} = \sqrt[12]{5}$

REMEMBER: The preceding examples only work when the bases are the same!

Note: $a^2 \cdot a^5 = a^7$; $a^2 \cdot b^7 = a^2 \cdot b^7$ (This cannot be simplified)

3. In a simplified radical expression, we should have only one radical. Therefore it is necessary to change all rational exponents in an expression to a common denominator so that they have the same root index.

(a) $5^{2/3} \cdot x^{3/5} = 5^{10/15} \cdot x^{9/15} = \sqrt[15]{5^{10}\, x^9}$

(b) $a^{1/3}\, b^{1/4}\, c^{1/2} = a^{4/12}\, b^{3/12}\, c^{6/12} = \sqrt[12]{a^4\, b^3\, c^6}$

(c) $\sqrt{x^3} \cdot \sqrt[5]{x^2} = x^{3/2}\, x^{2/5} = x^{15/10}\, x^{4/10} = x^{19/10} = x\sqrt[10]{x^9}$

(d) $\dfrac{\sqrt[3]{x^2}}{\sqrt[6]{y^5}} = \dfrac{x^{2/3}}{y^{5/6}} = \left(\dfrac{x^{2/3}}{y^{5/6}}\right)\left(\dfrac{y^{1/6}}{y^{1/6}}\right) = \dfrac{x^{2/3}\, y^{1/6}}{y} = \dfrac{\sqrt[6]{x^4\, y}}{y}$

EXERCISE 3 Write each of the following using a single radical.

1. $\sqrt[4]{x^2} \cdot \sqrt[5]{x^3}$

2. $\sqrt[3]{7x^2} \cdot \sqrt{y}$

3. $\dfrac{\sqrt[3]{w^5}}{\sqrt[5]{y^2}}$

4. $\sqrt{\sqrt{\sqrt{2}}}$

5. $\sqrt[3]{\sqrt[4]{\sqrt{x}}}$

6. $4\sqrt[3]{w^2} \cdot \sqrt[4]{w^5}$

7. $\sqrt{7} \div \sqrt[3]{p}$ 8. $\left[(5w)^{1/3}\right]^{2/5}$ 9. $\sqrt[3]{\sqrt[5]{32}}$

10. $(2^{4/5})^{3/4}$ 11. $7^{3/4} \div 7^{5/9}$ 12. $x^{3/5}\, y^{1/4}\, z^{3/10}$

SECTION IV RATIONALIZING THE DENOMINATOR WHEN IT IS A BINOMIAL

Sometimes the denominator presents more of a challenge than those we have seen so far. An example of this:

$$\frac{5}{\sqrt{a} + 2}$$

We have to get rid of the radical in the denominator, but we can't do it by multiplying by any one term.

Let us recall: $(a + b)(a - b) = a^2 - b^2$

This comes in handy now because if either 'a' or 'b' in the first binomial is a radical, then multiplying it by the second binomial would produce $a^2 - b^2$, thereby eliminating the radical...thereby rationalizing the denominator.

Let's see how this works:

Example 1: $$\frac{5}{(\sqrt{a} + 2)}\left(\frac{\sqrt{a} - 2}{\sqrt{a} - 2}\right) = \frac{5(\sqrt{a} - 2)}{(\sqrt{a})^2 - 2^2} = \frac{5\sqrt{a} - 10}{a - 4}$$

Example 2: $$\frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$$

Rationalizing the denominator:

$$\left(\frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}\right)\left(\frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}}\right) = \frac{5 - 2\sqrt{10} + 2}{5 - 2} = \frac{7 - 2\sqrt{10}}{3}$$

Note: We are using a very special expression to rationalize the denominator. We call it the conjugate expression.

$a + b$ is the conjugate of $a - b$

$\sqrt{5} - \sqrt{2}$ is the conjugate of $\sqrt{5} + \sqrt{2}$

$\sqrt{x} + 1$ is the conjugate of $\sqrt{x} - 1$

EXERCISE 4 Simplify each of the following by rationalizing the denominator.

1. $\frac{3}{3 - \sqrt{7}}$ 2. $\frac{1}{\sqrt{a} + \sqrt{b}}$ 3. $\frac{5}{\sqrt{6} - 2}$

4. $\frac{\sqrt{7}}{\sqrt{7} - 5}$ 5. $\frac{a}{\sqrt{5} - a}$ 6. $\frac{x + 1}{\sqrt{x} + 1}$

7. $\frac{\sqrt{w} - 2}{\sqrt{w} + 2}$ 8. $\frac{\sqrt{7} + \sqrt{2}}{\sqrt{7} - \sqrt{2}}$

SECTION V. SOLVING EQUATIONS WHICH CONTAIN RADICALS

A. The idea in solving an equation is to find out the numerical value of a variable. In other words, when we have:

$$x = \text{a number},$$

then we have solved the equation. If we have $\sqrt{x} = 7$, the equation is not solved. But, we can then square both sides of the equation (or raise both sides of the equation to the second power.)

$$(\sqrt{x})^2 = 7^2$$

$$x = 49$$

Now the equation is solved. Check for yourself. If $x = 49$, is it true that $\sqrt{x} = 7$?

Before using the squaring technique, be sure to isolate the radical, or it won't help. Study the following:

$$\sqrt{x} + 3 = 7$$

If we square both sides we get: $(\sqrt{x} + 3)^2 = 7^2$

$$x + 6\sqrt{x} + 9 = 49 \quad \text{(Not a help!)}$$

Squaring both sides before isolating the radical does not help because we still have the radical. If, however, you isolate the radical first by getting rid of the ' + 3', things are better. (To get rid of the ' + 3' we actually subtract 3 from both sides.)

$$\sqrt{x} + 3 = 7$$

$$\sqrt{x} = 4$$

$$(\sqrt{x})^2 = 4^2$$

$$x = 16$$ (Check this answer in the original equation.)

IMPORTANT NOTE: When solving an equation should you have a situation such as $-x = 7$, you may not stop. You must then multiply both sided of the equation by -1. Thus obtaining: $x = -7$.

AN EQUATION IS NOT SOLVED IF THE VARIABLE IS PRECEDED BY A NEGATIVE SIGN.

Let's try another example:

$$\sqrt{x+5} - 4 = 12$$

(1) Isolate the radical: $\sqrt{x+5} = 16$

(2) Square both sides: $x + 5 = 256$

(3) Solve: $x = 251$

(4) Check your answer: $\sqrt{251+5} - 4 \stackrel{?}{=} 12$

B. Sometimes we have more than one radical. In this case, we have to repeat the isolating process.

Example: $\sqrt{x-56} = \sqrt{x} - 4$

(1) Since one radical is already isolated--square

$$(\sqrt{x-56})^2 = (\sqrt{x} - 4)^2$$
$$x - 56 = x - 8\sqrt{x} + 16$$

(2) Combine and isolate the second radical $-72 = -8\sqrt{x}$

(3) Divide by -8 $9 = \sqrt{x}$

(4) Square both sides $81 = x$

(5) Check you answer:

$$\sqrt{81-56} \stackrel{?}{=} \sqrt{81} - 4$$
$$\sqrt{25} = 9 - 4$$
$$5 = 5 \quad \text{(Check)}$$

C. Squaring both sides will not always solve the problem. We sometimes have to raise both sides to a power other than two.

Example 1: $\sqrt[3]{x} + 2 = 5$

(1) Isolate the radical: $\sqrt[3]{x} = 3$

(2) Raise both sides to the third power: $(\sqrt[3]{x})^3 = 3^3$

(3) Simplify: $x = 27$

(4) Check $x = 27$ in the original equation.

Example 2: $x^{3/2} = 27$

(1) Raise both sides to the 2/3 power: $(x^{3/2})^{2/3} = (27)^{2/3}$

(2) Simplify: $x = 9$

(3) Check $x = 9$ in the original equation.

ALWAYS CHECK YOUR ANSWERS. The process of raising both sides of an equation to a power very often produces a root, or roots, which WILL NOT WORK-- "false roots."

Example 3: $\sqrt{x+5} + 12 = 9$

(1) Isolate the radical: $\sqrt{x+5} = -3$ (You could stop here! The answer to an even root cannot be negative.)

(2) Square both sides: $x + 5 = 9$

(3) Solve: $x = 4$

(4) Check: $\sqrt{4+5} + 12 \stackrel{?}{=} 9$ NO

Sol: $x = \emptyset$ (There is no real number solution.)

Example 4: $\sqrt{x+5} = x + 3$

(1) Square both sides: $x + 5 = x^2 + 6x + 4$

(2) Equate to zero: $0 = x^2 + 5x + 4$

(3) Factor: $0 = (x + 4)(x + 1)$

(4) Solve for x: $x = -4$ or $x = -1$

(5) Check both roots. Notice that $x = -4$ does not work. $\sqrt{1} \neq -1$
$x = -1$ checks. The solution is $x = -1$.

Now try these exercises on your own. (Remember to isolate the radical and check your answers.)

EXERCISE 5 For each of the following, solve for x:

1. $x^{2/3} = 4$
2. $x^{3/2} - 3 = 5$
3. $1 + \sqrt{a-3} = 4$
4. $\sqrt{2x+5} - 7 = -4$
5. $\sqrt{2a-3} + 7 = 10$
6. $\sqrt{x+2} + 4 = x$
7. $\sqrt[3]{x+2} + 5 = 8$
8. $\sqrt{x+10} = \sqrt{x-1} + 3$
9. $\sqrt{x-5} + 7 = 5$
10. $\sqrt{x-2} = x - 8$
11. $\sqrt{x-3} + 3 = x$
12. $\sqrt{7-9x} = 3 - x$

If you have done these exercises with a high degree of success, go on to try the Trial Run. If you need more practice, try the following exercises:

Dolciani, Algebra 2: pp. 328; 333-4; 336-7

Payne, Algebra 2: pp. 6; 9; 11; 13; 18-9; 357

ANSWERS

EXERCISE 1: 2. (a) 5; (b) 2; (c) 2; (d) 25; (e) 27; (f) -8; (g) 1,000; (h) -7; (i) ∅; (j) -3; (k) 1,000; (l) 32; (m) 1/10; (n) 1/8 (Note the difference between f and n; (o) 1/27

3. (a) $\sqrt[3]{x}$ (b) $\sqrt[3]{y^2}$ (c) $\sqrt{w}$ (d) $2\sqrt[4]{a^3}$

(e) $\sqrt[4]{(2a)^3}$ (f) $\sqrt[4]{2a^3}$ (g) $\sqrt[3]{6x^2}$ (h) $\sqrt[5]{ab^2}$

(i) $2\sqrt{xy}$ (j) $c\sqrt[6]{a^5 b}$ (k) $xy\sqrt[4]{z}$ (l) $\sqrt{a} + \sqrt{b}$

(m) $\sqrt[3]{x} - \sqrt[3]{y}$ (n) $6\sqrt{a} + 7\sqrt{b}$

4. (a) $w^{1/2}$ (b) $x^{1/3}y^{1/3}$ or $(xy)^{1/3}$ (c) $a^{2/5}b^{3/5}$

(d) $(pq)^{1/2}$ (e) $p^{1/5}q^{2/5}$ (f) $(5w^2)^{1/7}$ (g) $(5w)^{2/7}$

(h) $5w^{2/7}$ (i) $5^{1/7}w^2$ (j) $a\,b^{1/2}c^{1/2}$

(k) $a^{1/3}w^{5/3}$ (l) $x^{2/7}y^{3/7}z^{9/7}$ (m) $a^{1/3}b^{1/2}$

EXERCISE 2: 1. $5\sqrt[3]{5}$ 2. $2y^2\sqrt[5]{x^3}$ 3. $7xy\sqrt{y}$ 4. $3ab^3c\sqrt[4]{c^3}$

5. $\sqrt{5}$ 6. $p\sqrt[5]{16}$ 7. 2 8. $\frac{2}{x}$ 9. $3\sqrt[3]{12}$

10. $32\sqrt[3]{2}$ 11. $\sqrt[3]{3x^2}$ 12. $\frac{\sqrt[3]{25}}{256}$ 13. $5xy^2\sqrt[4]{x}$

14. $x\sqrt{5x}$ 15. $\frac{\sqrt[3]{2x}}{y}$ 16. $\frac{\sqrt{5x}}{x^2}$ 17. $\frac{3\sqrt[3]{x}}{x}$

18. $\frac{2\sqrt[5]{2a^3b^3}}{b}$ 19. $\frac{\sqrt[4]{x}}{y^3}$ 20. $\frac{7\sqrt[5]{x}}{x}$ 21. $\frac{\sqrt[7]{3x^5y^5z^6}}{xyz}$

22. $125\sqrt[7]{125}$ 23. $\left(\frac{2}{3}\right)^8$ 24. $\frac{3x^2\sqrt[3]{3yz}}{y}$ 25. $\frac{2\sqrt[5]{2y^3}}{x}$

26. $\frac{\sqrt[6]{8b^2a^4}}{a^2}$ 27. $\frac{10\sqrt{2xy}}{xy}$ 28. $5x^3$ 29. $\frac{2\sqrt[5]{wy^2z^4}}{yz}$

30. $\frac{2z^2\sqrt[3]{5y}}{xy}$

EXERCISE 3

1. $x\sqrt[10]{x}$
2. $\sqrt[6]{7^2 x^4 y^3}$
3. $\frac{w\sqrt[15]{w^{10} y^9}}{y}$
4. $\sqrt[8]{2}$
5. $\sqrt[24]{x}$
6. $4w\sqrt[12]{w^{11}}$
7. $\frac{\sqrt[6]{7^3 p^4}}{z}$
8. $\sqrt[15]{25w^2}$ or $\sqrt[15]{(5w)^2}$ NOT $\sqrt[15]{5w^2}$
9. $\sqrt[3]{2}$
10. $\sqrt[5]{8}$
11. $\sqrt[36]{7^7}$
12. $\sqrt[20]{x^{12} y^5 z^6}$

EXERCISE 4:

1. $\frac{9 + 3\sqrt{7}}{2}$
2. $\frac{\sqrt{a} - \sqrt{b}}{a - b}$
3. $\frac{5\sqrt{6} + 10}{2}$
4. $\frac{7 + 5\sqrt{7}}{-18}$
5. $\frac{a^2 + a\sqrt{5}}{5 - a^2}$
6. $\frac{(x + 1)(\sqrt{x} - 1)}{x - 1}$
7. $\frac{w - 4\sqrt{w} + 4}{w - 4}$
8. $\frac{9 + 2\sqrt{14}}{5}$

EXERCISE 5:

1. 8
2. 4
3. 12
4. 2
5. 6
6. 7 (Reject 2)
7. 25
8. $\frac{10}{9}$
9. ∅ (reject 9)
10. 11 (reject 6)
11. 4; 3
12. -1; -2

ALGEBRA 2 RATIONAL EXPONENTS TRIAL RUN

I. Evaluate

1. $\sqrt[6]{64} =$ _____ 2. $81^{1/4} =$ _____ 3. $\sqrt[3]{1{,}000} =$ _____

4. $25^{1/2} =$ _____ 5. $-25^{1/2} =$ _____ 6. $(-25)^{1/2} =$ _____

7. $-8^{1/3} =$ _____ 8. $(-8)^{1/3} =$ _____ 9. $16^{3/2} =$ _____

10. $64^{-2/3} =$ _____ 11. $1{,}000^{2/3} =$ _____ 12. $\left(\frac{8}{27}\right)^{-2/3} =$ _____

13. $-1{,}000{,}000^{-5/6} =$ _____ 14. $81^{3/4} =$ _____ 15. $(-32)^{3/5} =$ _____

II. Simplify. (Check your answers--before using ours--to make sure that you have eliminated all the "no-nO's." (See L.A.P.))

1. $\sqrt[3]{81}$ 2. $\sqrt[6]{64x^3y^6}$ 3. $\sqrt[3]{16x^{-3}}$

4. $\sqrt[6]{8}$ 5. $\sqrt[n]{5^{2n}}$ 6. $\sqrt{75a^3b^2}$

7. $\sqrt[4]{25p^{-4}q^{-3}}$ 8. $\sqrt{98x^9}$ 9. $\sqrt[4]{\left(\frac{25}{4}\right)^8}$

10. $16^{5/3}$ 11. $\sqrt{\frac{64}{x}}$ 12. $\sqrt[3]{ab^{-6}}$

13. $\sqrt[7]{81^3}$ 14. $\sqrt[3]{24x^{-2}y^{-6}}$ 15. $\sqrt[n]{3^{2n}x^{-3n}}$

16. $\sqrt[3]{\frac{81a}{b^3c}}$ 17. $\sqrt[5]{64x^7y^{10}z^{-15}}$ 18. $\sqrt{\frac{50}{abc}}$

19. $\sqrt[4]{32x^{-3}y^8}$ 20. $18x^{2/3}$

III. Simplify each of the following. Be sure all your answers are in a form involving only one radical.

1. $\sqrt[3]{x^2}\cdot\sqrt[5]{x^2}$ 2. $\sqrt[4]{a^3}\cdot\sqrt{a}\cdot\sqrt[3]{a^2}$ 3. $\sqrt{\sqrt{\sqrt{w}}}$

4. $\sqrt[15]{\sqrt[3]{\sqrt{7}}}$ 5. $\sqrt[7]{5x^2y}\cdot\sqrt[5]{z^3}$ 6. $\frac{\sqrt[4]{a^5}}{\sqrt{3}}$

7. $\frac{\sqrt[6]{p^5}}{\sqrt[5]{p^2}}$ 8. $\frac{\sqrt[3]{a^5}}{\sqrt[4]{p}}$ 9. $\left[(6a^2)^{3/5}\right]^{1/4}$

10. $\sqrt[4]{\sqrt[5]{64}}$ 11. $(2^{2/3})^{5/2}$ 12. $3^{4/5}\div 3^{3/7}$

Over

13. $a^{1/4}\, b^{2/3}\, c^{5/6}$ 14. $5\sqrt[3]{5^2}\cdot\sqrt[4]{125}$ 15. $\sqrt{11} \div \sqrt[5]{w^2}$

IV Rationalize the denominators and simplify:

1. $\frac{2}{\sqrt{5}+a}$
2. $\frac{1}{\sqrt{c}-\sqrt{d}}$
3. $\frac{3}{\sqrt{7}-2}$
4. $\frac{\sqrt{5}}{\sqrt{5}-1}$
5. $\frac{w}{\sqrt{7}-w}$
6. $\frac{x+3}{\sqrt{x}+3}$
7. $\frac{\sqrt{p}+5}{\sqrt{p}-5}$
8. $\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$

V. Solve and give all valid real roots:

1. $x^{3/4} = 125$
2. $x^{5/2} + 5 = 37$
3. $\sqrt{x} + 26 = 35$
4. $\sqrt{x+2} - 3 = 7$
5. $\sqrt{2a-5} + 9 = 12$
6. $\sqrt{x+5} = x - 7$
7. $\sqrt{3x-5} - \sqrt{3x} = -1$
8. $5\sqrt{x} = \sqrt{3x-2}$
9. $5\sqrt{3a+4} - 20 = 5$
10. $\sqrt{3a+10} = \sqrt{2a-1} + 2$

ANSWERS

I. 1. 2 2. 3 3. 10 4. 5 5. −5 6. ∅ 7. −2 8. −2 9. 64 10. $\frac{1}{16}$ 11. 100 12. $\frac{9}{4}$ 13. $\frac{1}{100,000}$ 14. 27 15. −8

II. 1. $3\sqrt[3]{3}$ 2. $2y\sqrt{x}$ 3. $\frac{2\sqrt[3]{2}}{x}$ 4. $\sqrt{2}$ 5. 25 6. $5ab\sqrt{3a}$ 7. $\frac{\sqrt[4]{25q}}{pq}$ 8. $7x\sqrt[4]{2x}$ 9. $\frac{625}{16}$ 10. $64\sqrt[3]{4}$ 11. $\frac{8\sqrt{x}}{x}$ 12. $\frac{\sqrt[3]{a}}{b^2}$ 13. $3\sqrt[7]{3^5}$ 14. $\frac{2\sqrt[3]{3x}}{xy^2}$ 15. $\frac{9}{x^3}$ 16. $\frac{3\sqrt[3]{3ac^2}}{bc}$

Trial Run Answers continued

17. $\frac{2xy^2 \sqrt[5]{2x^2}}{z^3}$ 18. $\frac{5\sqrt{2abc}}{abc}$ 19. $\frac{2y^2\sqrt[4]{2x}}{x}$

20. $18\sqrt[3]{x^2}$

III. 1. $x\sqrt[15]{x}$ 2. $a\sqrt[12]{a^{11}}$ 3. $\sqrt[8]{w}$ 4. $\sqrt[90]{7}$

5. $\sqrt[35]{5^5 x^{10} y^5 z^{21}}$ 6. $\frac{a\sqrt[4]{9a}}{3}$ 7. $\sqrt[30]{p^{13}}$ 8. $\frac{a\sqrt[12]{a^8 p^9}}{p}$

9. $\sqrt[20]{6^3 a^6}$ 10. $\sqrt[10]{8}$ 11. $2\sqrt[3]{4}$ 12. $\sqrt[35]{3^{13}}$

13. $\sqrt[12]{a^3 b^8 c^{10}}$ 14. $25\sqrt[12]{5^5}$ 15. $\frac{\sqrt[10]{11^5 w^6}}{w}$

IV. 1. $\frac{2\sqrt{5} - 2a}{5 - a^2}$ 2. $\frac{\sqrt{c} + \sqrt{d}}{c - d}$ 3. $\sqrt{7} + 2$

4. $\frac{5 + \sqrt{5}}{4}$ 5. $\frac{\sqrt{7}w + w^2}{7 - w^2}$ 6. $\frac{(x+3)(\sqrt{x} - 3)}{x - 9}$

7. $\frac{p + 10\sqrt{p} + 25}{p - 25}$ 8. $\frac{5 - \sqrt{21}}{2}$

V. 1. 625   2. 4   3. 81   4. 98

5. 7   6. 11 (reject 4)   7. 3   8. $\emptyset$ (reject $-\frac{1}{11}$)

9. 7   10. 13, 5